ISYE 8803 HW7

1. Compress sensing

   Suppose that $\boldsymbol{A}$ is a 256×512 dimensional matrix. Let $\boldsymbol{x} \in \mathbb{R}^{512}$ be an s-sparse vector, and $\boldsymbol{y} = \boldsymbol{Ax}$ the observed system output. This problem is concerned with $\ell_1$minimization in recovering $\boldsymbol{x}$, i.e.

   $$\min_{\boldsymbol{x}} \ \|\boldsymbol{x}\|_1 \quad \text{s.t.} \quad \boldsymbol{Ax} = \boldsymbol{y} \qquad (1)$$

   Let s range between 1 and 128. For each choice of s, run 20 independent numerical experiments: in each experiment, generate $\boldsymbol{A}$ such that its elements are i.i.d standard Gaussian random variables, generate $\boldsymbol{x}$ as a random s-sparse signal (e.g. you may generate the support of $\boldsymbol{x}$ uniformly at random, with each non-zero entry drawn from the standard Gaussian distribution), and solve (1). An experiment is claimed successful if the solution $\boldsymbol{x}$ returned by (1) obey $\|\hat{x} - x\|_2 \leq 10^{-4}\|x\|_2$. Report the empirical success rates (averaged over 20 experiments) for each choice of s.
2. Smooth-sparse decomposition.

Yan, et al. [1] formulated the following optimization problem for smooth sparse decomposition.

$$\min_{\theta,\theta_a} \|Y - B\theta - B_a\theta_a\|_2^2 + \lambda\theta^\top\Omega\theta + \gamma\|\theta_a\|_1$$

Here $\Omega = D^\top D$, where D is the difference matrix.

   a) Argue that the formulation is a convex problem.
   b) Download the data file `data.mat` and plot smooth part, anomaly part and error part of the data are separated in your generated plot.
3. Matrix recovery
   (a) Let M_0 be a random 50×50 matrix with rank 2. Randomly select 50% of its entries and replace them with 0.
   (b) Solve the following optimization problem

   $$\min_{\mathrm{M}} \|M\|_*$$

   $$\text{subject to} \quad \mathrm{M}(\mathrm{i},\mathrm{j}) = \mathrm{M}_0(i,j) \quad \forall (i,j) \in \{known\ set\}$$

   (c) Report the relative reconstruction error defined as:

   $$\frac{\|M - M_0\|_F}{\|M_0\|_F}$$

4. RKHS Ridge Regression: A noisy smooth image is given in "peaks.mat". The goal is to denoise the image using RKHS.
   a) Create a 2D Gaussian kernel basis (Gram matrix) by finding the Kronecker product of two 1D Gaussian kernels with bandwidth 1.
   b) Use these Gram matrices to estimate the value of each pixels using the information of other pixels following the RKHS Ridge Regression procedure.
   c) Compute and plot the smooth image and estimate the noise standard deviation.