Deterministic Optimization

Introduction

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Introduction to Optimization

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Learning Objectives

- Define optimization
- Formulate optimization problems

What is optimization?

Oxford: The action of making the best or most effective use of a situation or resource.

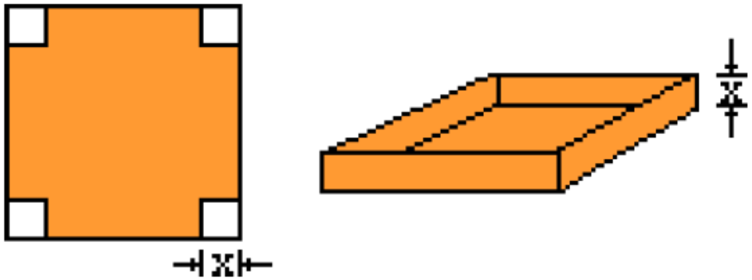
Merriam-Webster: an act, process, or methodology of making something (such as a design, system, or decision) as fully perfect, functional, or effective as possible; specifically : the mathematical procedures (such as finding the maximum of a function) involved in this

Our definition:

A mathematical approach for seeking a “best” “decision/action” from a “set of alternatives”

Example: Designing a box

Problem: Given a 1'X1' cardboard, cut out corners and fold to make a box of maximum volume.



- Decision: x = How much to cut?
- Alternatives: $0 \leq x \leq 1/2$
- Best: Maximize volume $V(x) = x(1-2x)^2$

Optimization formulation:

$$\begin{array}{ll} \max & x(1-2x)^2 \\ \text{s.t.} & 0 \leq x \leq 1/2 \end{array}$$

Example: Data fitting

Problem: Given N data points $(y_1, x_1)..(y_N, x_N)$ where $y_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^n$ for all $i = 1..N$, find a line $y = a^\top x + b$ that best fits the data.

- Decision: A vector $a \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$
- Set of alternatives: All n-dimensional vectors and scalars
- Best: Minimize sum of squared errors

Optimization Formulation:

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} (y_i - a^\top x_i - b)^2 \\ \text{s.t.} \quad & a \in \mathbb{R}^n, \; b \in \mathbb{R} \end{aligned}$$

Example: Product Mix

Problem: A firm make n different products using m types of resources. Each unit of product i generates p_i dollars of profit, and requires r_{ij} units of resource j. The firm has u_j units of resource j available. How much of each product should the firm make to maximize profits?

- Decision: How much of each product to make
- Alternatives: Defined by the resource limits
- Best: Maximize profits

Optimization formulation:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} r_{ij} x_i \leq u_j \quad && \forall\ j = 1, \ldots, m \\ & x_i \geq 0 && \forall\ i = 1, \ldots, n \end{aligned}$$

Example: Product Mix (contd.)

To compactly express this formulation, we can use vector-matrix notation.

- Decision vector: $$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
- Data vectors/matrix:

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}$$

Example: Product Mix (contd.)

Sum notation:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} r_{ij} x_i \leq u_j \quad && \forall\ j = 1, \ldots, m \\ & x_i \geq 0 && \forall\ i = 1, \ldots, n \end{aligned}$$

Matrix notation:

$$\begin{aligned} \max \quad & p^\top x \\ \text{s.t.} \quad & Rx \leq u \\ & x \geq 0 \end{aligned}$$

Important to distinguish between notation for data and variables

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Example: Project investment

Problem: A firm is considering investment in n different R&D projects. Project j requires an investment of c_j dollars and promises a return of r_j dollars. The firm has a budget of B dollars. Which projects should the firm invest in?

- Decision: Whether-or-not to invest in project
- Alternatives: Defined by budget
- Best: Maximize return on investment

Example: Project Investment (contd.)

Sum notation:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} r_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} c_j x_j \leq B \\
& x_j \in \{0,1\} \ \forall \ j = 1, \ldots, n
\end{aligned}
$$

Matrix notation:

$$
\begin{aligned}
\max \quad & r^\top x \\
\text{s.t.} \quad & c^\top x \leq B \\
& x \in \{0,1\}^n
\end{aligned}
$$

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Formulation Steps

- Encode decisions/actions as **decision variables** whose values we are seeking
- Identify the relevant **problem data**
- Express **constraints** on the values of the decision variables as mathematical relationships (inequalities) between the variables and problem data
- Express the **objective function** as a function of the decision variables and the problem data

Summary

- Optimization is a mathematical approach for finding a best option from a set of alternatives
- Formulation of an optimization problem requires identifying decision variables, constraints, objective function and problem data