# ISyE 6669
# Midterm Practice

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**Problem 1.** Consider the matrix $A = \begin{bmatrix} 0 & 3 & -2 & -1 \\ 4 & -9 & 1 & 3 \\ -6 & -15 & -1 & 5 \end{bmatrix}$. The rank of $A$ is

A 1

B 2

C 3

D 4

**Problem 2.** Suppose matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$ both have full row rank. Then the rank of $A + B$ is

A $m$

B $n$

C $\min\{m, n\}$

D Depends on $A$ and $B$

**Problem 3.** Consider the function $f(x, y) = x^3 - 2xy + y^2$. The first order Taylor's expansion of this function at $(-2, 3)$ is

A $6x + 2y - 13$

B $5x + 3y - 6$

C $13x - 3y - 6$

D $6x + 10y - 5$

**Problem 4.** The set $X = \{x \in [-1, 1]^5 : \sum_{i=1}^{5} x_i \geq -2\}$ is

A Unbounded and closed

B Bounded and not closed

C Bounded and closed

D Unbounded and not closed

**Problem 5.** Which set is convex

A $\{x \in \mathbb{R}^2 | \min\{x_1, x_2\} \geq 1\} + \{x \in \mathbb{R}^2 | x_1 \leq -1, x_2 \leq -2\}$

B $\{x \in \mathbb{R}^2 | x_1 \geq \ln x_2,\ x_2 > 0\}$

C $\{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \geq 4\}$

D $\{x \in \mathbb{R}^2 | \max\{x_1, x_2\} \geq 1\}$

**Problem 6.** Which function is convex on the indicated domain

A $f(x_1, x_2) = 2^{|x_1+1|+|x_2-x_1|}$ on $\mathbb{R}^2$

B $f(x) = -2x^2 + 6x + 10$ on $\mathbb{R}$

C $f(x) = x^5$ on $[-1, 1]$

D $f(x) = \ln(x^2)$ on $(0, +\infty)$

**Problem 7.** Which statement is TRUE?

A A function with convex level sets is always convex.

B An optimal solution of maximizing a convex function over a compact set lies on the boundary of the set.

C A non-convex optimization problem always has a unique optimal solution.

D A convex function cannot be a concave function at the same time.

**Problem 8.** Which of the following is a convex optimization problem

A $\min\{x^4 + y^4 : x^2 + y^2 \geq 20\}$

B $\max\{\cos(x) : x \in [0, 2\pi]\}$

C $\min\{\ln(x) : x \geq 1\}$

D $\min\{-x - 8y : x + 7y \geq 10, e^x \leq 1\}$

**Problem 9.** What is the outcome of the problem $\max\{412x - 511y \ : \ 0 \leq x \leq 1, y \leq 1\}$

A Infeasible

B Unbounded optimum

C The unique optimal solution is $x = 1, y = 1$.

D None of above

**Problem 10.** I have solved an optimization problem and got an optimal solution $x^*$. Suppose now a new constraint is added to my problem. If I find that $x^*$ satisfies the new constraint, then $x^*$ is an optimal solution of the modified problem.

A True

B False

C It depends on whether the optimization problem is maximization or minimization.

**Problem 11.** If I maximize a univariate convex function over a nonempty closed and bounded interval, then there has to be an optimal solution which is one of the end points.

A True

B False

**Problem 12.** Consider an unconstrained maximization problem $\max f(x)$. Suppose at a point $x_0$ we know $\nabla f(x_0) = 0$ and $\nabla^2 f(x_0)$ is positive definite. Then

A $x_0$ is a local maximizer

B $x_0$ is a local minimizer

C $x_0$ may be neither a local maximizer nor a local minimizer

D Whether $x_0$ is a local maximizer or minimizer depends on whether $f(x)$ is a convex function or not.

**Problem 13.** The gradient vector of $f(x_1, x_2) = \sqrt{x_1 - 1} + (x_2 - 1)^{1/4}$ at point $(x_1, x_2) = (3, 2)$

A Not defined

B $\left(\frac{1}{2\sqrt{2}}, \frac{1}{4}\right)^\top$

C $\left(\frac{1}{\sqrt{2}}, 1\right)^\top$

D $\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)^\top$

**Problem 14.** The Hessian matrix of $f(x_1, x_2) = 2(x_1 - 3)^2 + 6x_1x_2 + 2(x_2 + 3)^2 + 6x_1 + 2x_2$ at point $(x_1 = 2, x_2 = 3)$ is

A $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

B $\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$

C $\begin{bmatrix} 4 & 8 \\ 8 & 2 \end{bmatrix}$

D $\begin{bmatrix} 4 & 4 \\ 4 & 1 \end{bmatrix}$

**Problem 15.** Consider the problem min $\{\boldsymbol{c}^\top \boldsymbol{x} : \boldsymbol{x} \in X\}$ where $X$ is a nonempty compact set in $\mathbb{R}^n$. Suppose $\boldsymbol{c} \neq \mathbf{0}$. What can you say about the problem:

A The problem may have an unbounded optimal solution

B An optimal solution can be in the interior of $X$

C The problem can be infeasible

D The optimal solution exists and is unique

**Problem 16.** Which of the following is not a feasible point of $X = \{x \in \mathbb{R}^2 \mid x_1 - x_2 \leq -1, 2x_1 - x_2 \leq 0, x_2 \geq -\frac{1}{2}\}$?

A $(0, -\frac{1}{2})$

B $(\frac{1}{2}, \frac{3}{2})$

C $(0, 1)$

D $(-1, 0)$

**Problem 17.** For the transportation problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} \geq d_j, & j = 1, 2, ..., n \\ & \sum_{j=1}^{n} x_{ij} \leq s_i, & i = 1, 2, ..., m \\ & x_{ij} \geq 0 & i = 1, 2, ..., m, j = 1, 2, ..., n \end{aligned}$$

Which of the following situations must cause infeasibility?

A The total supply $\sum_{i=1}^{m} s_i$ is greater than the total demand $\sum_{j=1}^{n} d_j$

B There exists a demand $d_j$ which is greater than the total supply $\sum_{i=1}^{m} s_i$

C The total supply $\sum_{i=1}^{m} s_i$ is equal to the total demand $\sum_{j=1}^{n} d_j$

D Some $s_i$ is smaller than all demand $d_j$ for $j = 1, 2, ..., n$.

**Problem 18.** Which of the following functions can be reformulated as convex piecewise linear functions?

A $f(x) = \min\{10x - 1,\ 2x + 10,\ 20\}, \quad x \in \mathbb{R}$

B $f(x) = \sum_{i=1}^{2019} 2^{-i} |e^{i/2} x - \sin(i)|, \quad x \in \mathbb{R}$

C $f(x_1, x_2) = -|x_1| + |x_2|, \quad -1 \leq x_1 \leq 2, 0 \leq x_2 \leq 2.$

D None of the above.

**Problem 19.** Choose the correct statement about the "here-and-now" decision and the "wait-and-see" decision.

A The "wait-and-see" decision can be made before part of the uncertainty is realized.

B The "here-and-now" decision may change after the uncertainty is realized.

C The "here-and-now" decision must be made before the uncertainty is realized.

**Problem 20.** Consider the following nonlinear optimization problem:

$$\begin{aligned} &\min\ |x-2| + 3|1-x| \\ &\text{s.t.}\ \ x \geq 0 \end{aligned}$$

Which of the following linear reformulations is equivalent to it?

A

$$\begin{aligned} \min\ & z_1 + 3z_2 \\ \text{s.t.}\ \ & z_1 \geq x-2 \\ & -z_1 \leq x-2 \\ & z_2 \geq 3(1-x) \\ & -z_2 \leq 3(1-x) \\ & x \geq 0 \end{aligned}$$

B

$$\begin{aligned} \min\ & z_1 + 3z_2 \\ \text{s.t.}\ \ & z_1 \geq x-2 \\ & -z_1 \geq x-2 \\ & z_2 \geq 1-x \\ & -z_2 \geq 1-x \\ & x \geq 0 \end{aligned}$$

C

$$\begin{aligned} \min\ & z_1 + 3z_2 \\ \text{s.t.}\ \ & z_1 \geq x-2 \\ & -z_1 \leq x-2 \\ & z_2 \geq 1-x \\ & -z_2 \leq 1-x \\ & x \geq 0 \end{aligned}$$

D

$$\begin{aligned} \min\ & -z_1 - 3z_2 \\ \text{s.t.}\ & z_1 \geq x - 2 \\ & -z_1 \leq x - 2 \\ & z_2 \geq 1 - x \\ & -z_2 \leq 1 - x \\ & x \geq 0 \end{aligned}$$

**Problem 21.** Consider the following nonlinear objective that is the average of the $\ell_1$ metric and $\ell_\infty$ metric

$$\min_x \left\{ \frac{1}{2}(|x-4| + |x+5|) + \frac{1}{2}\max\{|x-4|, |x+5|\} \right\}.$$

Which is the correct reformulation as a linear program?

A

$$\begin{aligned} \min\ & \frac{1}{2}(z_1 + z_2) + \frac{1}{2}(x-4) + \frac{1}{2}(x+5) \\ \text{s.t.} & -z_1 \leq x - 4 \\ & z_1 \leq x - 4 \\ & -z_2 \leq x + 5 \\ & z_2 \leq x + 5 \end{aligned}$$

B

$$\begin{aligned} \min\ & \frac{1}{2}(z_1 + z_2) + \frac{1}{2}z_3 + \frac{1}{2}z_4 \\ \text{s.t.} & -z_1 \leq x - 4 \leq z_1 \\ & -z_2 \leq x + 5 \leq z_2 \\ & -z_3 \leq x - 4 \leq z_3 \\ & -z_4 \leq x + 5 \leq z_4 \end{aligned}$$

C

$$\begin{aligned} \min\ & \frac{1}{2}z_1 + \frac{1}{2}z_2 \\ \text{s.t.} & -z_1 \leq x - 4 \leq z_1 \\ & -z_1 \leq x + 5 \leq z_1 \\ & -z_2 \leq x - 4 \leq z_2 \\ & -z_2 \leq x + 5 \leq z_2 \end{aligned}$$

D

$$\begin{aligned} \min \quad & \frac{1}{2}(z_1 + z_2) + \frac{1}{2}z_3 \\ \text{s.t.} & -z_1 \leq x - 4 \leq z_1 \\ & -z_2 \leq x + 5 \leq z_2 \\ & -z_3 \leq x - 4 \leq z_3 \\ & -z_3 \leq x + 5 \leq z_3 \end{aligned}$$

**Problem 22.** Which of the following statements about the set $P = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + 2x_2 - 3x_3 + 4x_4 - 5x_5 = 0\}$ in $\mathbb{R}^5$ is TRUE?

A $P$ is not a polyhedron, because a polyhedron should be defined by inequalities.

B $P$ is a polyhedron but not convex.

C $P$ is unbounded and is the intersection of two halfspaces, $\{\boldsymbol{x} \in \mathbb{R}^5 : x_1+2x_2-3x_3+4x_4-5x_5 \leq 0\} \cap \{\boldsymbol{x} \in \mathbb{R}^5 : x_1 + 2x_2 - 3x_3 + 4x_4 - 5x_5 \geq 0\}$.

D $P$ has an extreme ray $(3, 0, 1, 0, 0)^\top$.

**Problem 23.** If a standard form linear program is not degenerate and has a finite optimal solution, then the step size determined by the min-ratio test in each iteration of the simplex method must be strictly positive, and the objective function value must also be strictly improved at each iteration.

A True

B False

**Problem 24.** Find the extreme ray(s) of the polyhedron $P = \{(x, y) \in \mathbb{R}^2 : x - y \leq 1,\ x \geq 0,\ y \geq 0\}$.

A $[1, 1]^\top$

B $[1, 0]^\top$

C The polyhedron $P$ does not have any extreme ray.

**Problem 25.** Which of the following is TRUE?

A A polyhedron always contains at least one extreme ray.

B A nonempty polytope may not have a extreme point.

C The nonnegative orthant $\{(x_1, x_2, x_3) : x_i \geq 0, \forall i = 1, 2, 3\}$ has three extreme rays.

D A nonempty polyhedron in standard form may not have an extreme point.

**Problem 26.** Given a polyhedron $P$ in standard form where $A$ is a $4 \times 6$ matrix with full row rank, which of the following is not possible to be the number of basic feasible solutions of $P$?

A 20

B 15

C 1

D 0

**Problem 27.** Suppose a standard form LP has a nondegenerate basic feasible solution $\boldsymbol{x} = (x_1 = 1, x_2 = 2, x_3 = 0, x_4 = 2, x_5 = 0)$. Which of the following points cannot be a basic feasible solution adjacent to $\boldsymbol{x}$?

A $(2, 1, 3, 0, 0)$.

B $(0, 1, 0, 1, 3)$.

C $(0, 0, 0, 1, 1)$.

D $(1, 0, 2, 0, 1)$.

**Problem 28.** Suppose the direction to move in a simplex iteration is $\boldsymbol{d}^\top = \begin{pmatrix} -3 & 0 & -4 & 1 & 0 \end{pmatrix}$ and the current basic variables have values $(1, 2)$. Which variables are basic variables and what is the next iteration's basic feasible solution?

A Basic variables are $(x_2, x_5)$ and the next iteration's basic feasible solution is $\begin{pmatrix} 0 & 0 & -2 & 1 & 0 \end{pmatrix}^\top$.

B Basic variables are $(x_1, x_3)$ and the next iteration's basic feasible solution is $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}^\top$.

C Basic variables are $(x_1, x_3)$ and the next iteration's basic feasible solution is $\begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}^\top$.

**Problem 29.** If the the reduced costs of nonbasic variables $(x_3, x_4, x_5)$ are $(3, -1, 0)$ and the ratios are $(\frac{x_1}{-d_1}, \frac{x_2}{-d_2}) = (1, 0)$. Is this basic feasible solution degenerate and why?

A Yes, because currently the reduced cost of $x_5$ is 0.

B No, because no basic variable can be zero.

C Yes, because the basic variable $x_2$ must be 0.

**Problem 30.** Given an LP , write down an equivalent standard form LP.
$\max\{2x_1 - 3x_2 : x_1 + x_2 \leq 2,\ x_1 + 2x_2 \geq 3\}$

A $\max\{2x_1 - 3x_2\ :\ x_1 + x_2 + s_1 = 2,\ x_1 + 2x_2 - s_2 = 3, s_1 \geq 0, s_2 \geq 0\}$

B $\min\{-2x_1^+ + 2x_1^- + 3x_2\ :\ x_1^+ - x_1^- + x_2 + s_1 = 2,\ x_1^+ - x_1^- + 2x_2 - s_2 = 3, s_1 \geq 0, s_2 \geq 0, x_1^+ \geq 0, x_1^- \geq 0\}$

C $\min\{-2x_1^+ + 2x_1^- + 3x_2^+ - 3x_2^-\ :\ x_1^+ - x_1^- + x_2^+ - x_2^- + s_1 = 2,\ x_1^+ - x_1^- + 2x_2^+ - 2x_2^- - s_2 = 3, s_1 \geq 0, s_2 \geq 0, x_1^+ \geq 0, x_1^- \geq 0, x_2^+ \geq 0, x_2^- \geq 0\}$

D $\max\{2x_1^+ - 2x_1^- - 3x_2^+ + 3x_2^-\ :\ x_1^+ - x_1^- + x_2^+ - x_2^- + s_1 = 2,\ x_1^+ - x_1^- + 2x_2^+ - 2x_2^- - s_2 = 3, s_1 \geq 0, s_2 \geq 0, x_1^+ \geq 0, x_1^- \geq 0, x_2^+ \geq 0, x_2^- \geq 0\}$