W2M3L1 Knowledge Check

Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -3 \\ 2 & 2 & -2 \end{bmatrix}$. The rank of matrix A is

Solution

Use Gauss Elimination. This linearly combines rows/columns to arrive at upper diagonal form of original matrix. Start with

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -3 \\ 2 & 2 & -2 \end{bmatrix}.$$

Multiply row 1 by 2 and add to row 3:

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -3 \\ 0 & 2 & 0 \end{bmatrix}.$$

Multiply row 2 by -$\frac{1}{3}$ and add to row 1:

$$\begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ 3 & 1 & -3 \\ 0 & 2 & 0 \end{bmatrix}.$$

Multiply row 1 by 6 and add to row 3:

$$\begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ 3 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Rearrange (interchange rows):

$$\begin{bmatrix} 3 & 1 & -3 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

When in this form, rank is the number of rows with nonzero entries. Thus, the rank is $\boxed{2.}$

Another approach would have been to have recognized that column 3 is column 1 multiplied by -1 and column 2 is not a linear combination of column 1/3. Thus, due to column 3 being a scaled version of column 1, the matrix is one column removed from being of full rank. As full rank for a matrix with the same dimensions as A is 3, then the rank of A is $\boxed{2.}$

W2M3L2 Knowledge Check

Consider the function $f(x, y) = 2y + xy + x^2$. The first order Taylor's expansion of this function at (1, 1) is

Solution

Recall the first-order Taylor's approximation of a function at a point $\mathbf{x}_0$ is

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0).$$

We know that at $\mathbf{x}_0 = (x, y) = (1, 1)$,

$$f(\mathbf{x}_0) = 2(1) + (1)(1) + (1)^2 = 4.$$

Can solve for the gradient $\nabla f(\mathbf{x}_0)^{\mathrm{T}}$ as

$$\begin{aligned}\nabla f(\mathbf{x}_0) &= \left[\frac{\partial f(\mathbf{x})}{\partial x}\Big|_{\mathbf{x}=\mathbf{x}_0}, \frac{\partial f(\mathbf{x})}{\partial y}\Big|_{\mathbf{x}=\mathbf{x}_0}\right]^{\mathrm{T}} \\ &= [(y + 2x)|_{\mathbf{x}=\mathbf{x}_0}, (2 + x)|_{\mathbf{x}=\mathbf{x}_0}]^{\mathrm{T}} \\ &= [3, 3]^{\mathrm{T}}.\end{aligned}$$

Put all together to find

$$\begin{aligned}f(\mathbf{x}) &= 4 + \begin{bmatrix}3, 3\end{bmatrix}\begin{bmatrix}x - 1 \\ y - 1\end{bmatrix} \\ &= 4 + 3(x - 1) + 3(y - 1) \\ &= \boxed{-2 + 3x + 3y.}\end{aligned}$$

In order to calculate the second-order Taylor's approximation of a function at a point $\mathbf{x}_0$, we can use the formula given in lecture,

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}} \nabla^2 f(\mathbf{x}_0)(\mathbf{x}_0 - \mathbf{x}_0).$$

Here, we additionally need to compute the Hessian matrix, which for this problem is defined as

$$\begin{bmatrix}\frac{\partial^2 f(\mathbf{x})}{\partial x^2}\Big|_{\mathbf{x}=\mathbf{x}_0} & \frac{\partial^2 f(\mathbf{x})}{\partial x \partial y}\Big|_{\mathbf{x}=\mathbf{x}_0} \\ \frac{\partial^2 f(\mathbf{x})}{\partial y \partial x}\Big|_{\mathbf{x}=\mathbf{x}_0} & \frac{\partial^2 f(\mathbf{x})}{\partial y^2}\Big|_{\mathbf{x}=\mathbf{x}_0}\end{bmatrix}.$$

Evaluating using the given function, we find the Hessian matrix to be $\begin{bmatrix}2 & 1 \\ 1 & 0\end{bmatrix}$. We evaluate the second-oder Taylor's approximation as

$$\begin{aligned}f(\mathbf{x}) &= 4 + \begin{bmatrix}3, 3\end{bmatrix}\begin{bmatrix}x - 1 \\ y - 1\end{bmatrix} + \frac{1}{2}[x - 1, y - 1]\begin{bmatrix}2 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}x - 1 \\ y - 1\end{bmatrix} \\ &= 4 + 3(x - 1) + 3(y - 1) + x^2 + xy - 3x - y + 2 \\ &= x^2 + xy + 2y.\end{aligned}$$

W2M3L3 Knowledge Check

The set $X = \{(x, y) : x^2 + y^2 \geq 1\}$ is

Solution

Let us draw the region represented by the set. It is shown in Figure 1.

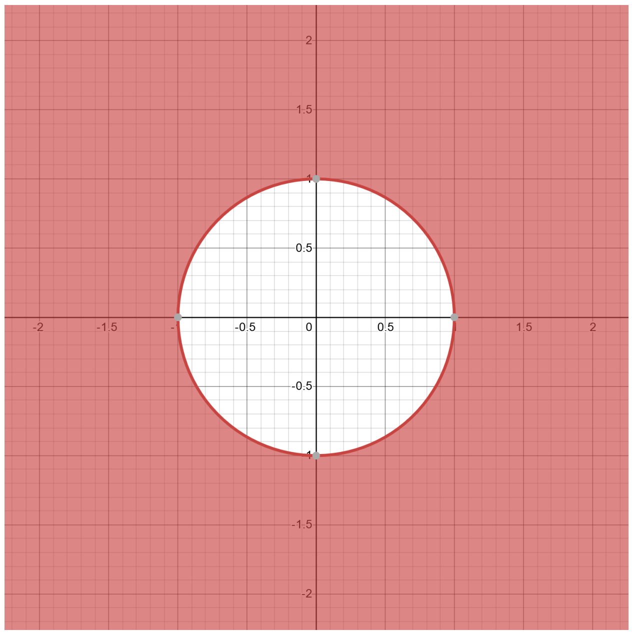

Figure 1: Geometric representation of the set X.

Recall that a set is closed if it includes its boundary points. As the given set includes the boundary points of the circle of radius 1 centered at the origin, it is a closed set.

Recall that a set is bounded if it can be enclosed in a large enough (hyper)-sphere or box. As the given set includes the points outside of the circle of radius 1 centered at the origin, and extends to infinity in all directions, the set is unbounded.

Thus, X is $\boxed{\text{closed but not bounded.}}$

W2M4L1 Knowledge Check

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

Solution

This is defined in lecture., Thus the solution is

$$\boxed{f\left(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\right) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \; \forall \, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and } \lambda \in [0,1].}$$

W2M4L2 Knowledge Check

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex function. Which of the following sets must be a convex set?

Solution

Consider $X = \{x : f(x) \geq 1\}$ with $f(x) = x^2$. This is the α upper level set. As can be seen, the α upper level set is shown in Figure 2. The level set is discontinuous, and the set is consequently not convex as a line segment connecting a point in one continuous region with a point in the other continuous region does not lie entirely in the set. However, an α lower level set would be convex.

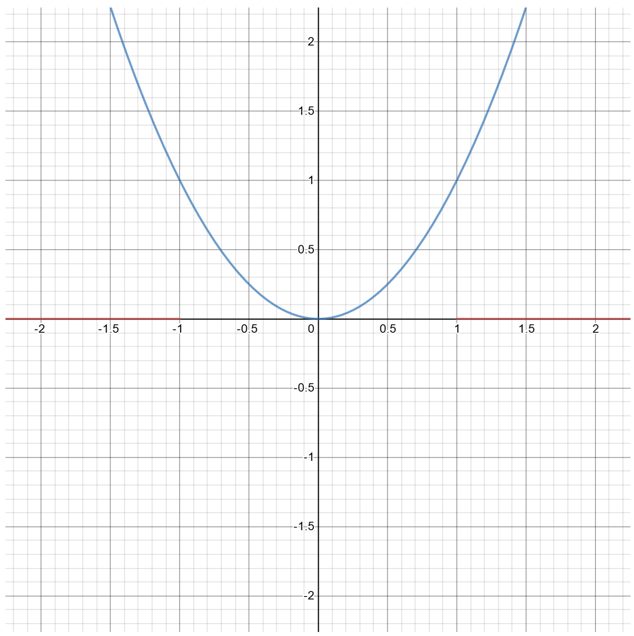

Figure 2: Representation of $X = \{x : f(x) \geq 1\}$ with $f(x) = x^2$.

Consider $X = \{(y, x) : y = f(x)\}$ with $f(x) = x^2$, which is shown in Figure 2. This set is exactly the quadratic line. A line segment connecting two points in that set does not lie entirely in the set and the set is not convex.

Consider $X = \{(y, x) : y \geq f(x)\}$. As described in lecture, this is the epigraph of the function f. Indeed, the epigraph of a convex function is a convex set. Thus, the correct answer is $\boxed{X = \{(y, x) : y \geq f(x)\}.}$

W2M4L3 Knowledge Check

Which of the following is a convex optimization problem?

Solution

Consider $\min\{x^2 + y^2 : x + y \geq 1\}$. The objective function $x^2 + y^2$ is a convex function, as it is the sum of two convex functions, x^2 and y^2. The constraint set $x + y \geq 1$ is a convex set. Thus, this is a convex optimization problem.

Consider $\max\{x + y : x^2 + y^2 \geq 1, 0 \leq x \leq 10, 0 \leq y \leq 10\}$. The objective function $x + y$ is an affine function, so whether the problem is a minimization problem or a maximization problem, this does not prevent it from being a convex optimization problem. The constraints define the set defined by the shaded region in Figure 3. This is not a convex set as a line segment from $(x, y) = (0, 1)$ to $(1, 0)$ is not contained within the set. Thus, the problem is not a convex optimization problem.

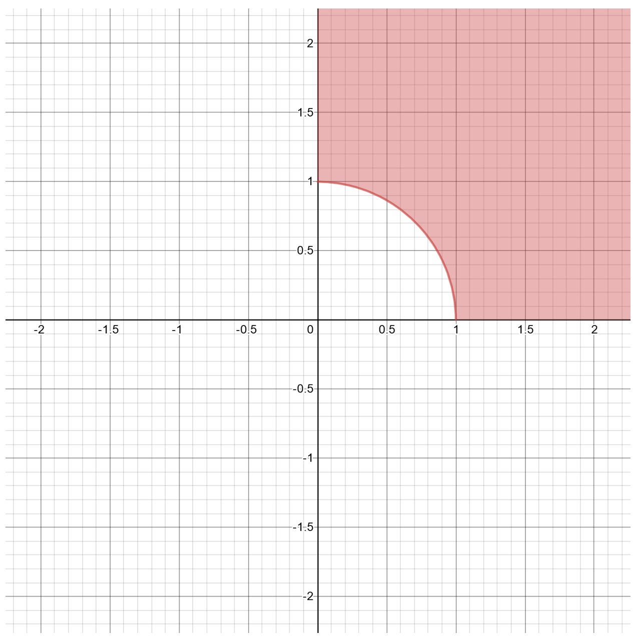

Figure 3: Representation of $X = \{(y, x) : x^2 + y^2 \geq 1, 0 \leq x \leq 10, 0 \leq y \leq 10\}$.

Consider $\min\{-x^2 : -1 \leq x \leq 1\}$. The objective function $-x^2$ is not a convex function, so the problem is not a convex optimization problem.

As determined above, the convex optimization problem is $\boxed{\min\{x^2 + y^2 : x + y \geq 1\}.}$