LECTURE 7
LINEAR PROGRAMMING I

AGENDA

- McDonald's diet problem
  - Formulation
  - Excel solution
- Product mix problem
  - Formulation
  - Excel solution
- Linear Programming: terminology

MCDONALD'S DIET PROBLEM

- You want your diet to meet some nutritious standards. According to your daily diet plan, you need to have:
  - at least 100 percent of the U.S. Recommended Dietary Allowance[1] (RDA) of vitamin C & calcium,
  - at least 55 grams of protein,
  - and at most 2000 calories.
- You are wondering if this can be accomplished by eating at McDonald's. Can you design the least-cost McDonald's daily meal plan that meets your daily nutritious standards?

Menu Item	Price (USD)	Calories	Protein	Fat	Sodium	Vit. A	Vit. C	Calcium	Iron
			(g)		(mg)	% U.S. RDA			
Hamburger	0.69	260	13	9	530	2	2	15	15
Big Mac	1.99	560	25	30	1010	8	2	25	25
Chicken McNuggets (6pcs)	1.99	250	15	15	670	2	2	2	4
Garden Salad	2.05	35	2	0	20	120	40	4	6
Baked Apple Pie	0.79	260	3	13	200	0	40	2	6

[1] National Institutes of Health (https://ods.od.nih.gov/Health_Information/Dietary_Reference_Intakes.aspx)

MCDONALD'S DIET PROBLEM FORMULATION

- Data for McDonald's Diet Problem

Menu Item	Price (USD)	Calories	Protein	Fat	Sodium	Vit. A	Vit. C	Calcium	Iron
			(g)		(mg)	% U.S. RDA			
Hamburger	0.69	260	13	9	530	2	2	15	15
Big Mac	1.99	560	25	30	1010	8	2	25	25
Chicken McNuggets (6pcs)	1.99	250	15	15	670	2	2	2	4
Garden Salad	2.05	35	2	0	20	120	40	4	6
Baked Apple Pie	0.79	260	3	13	200	0	40	2	6

1. What must be decided?
   - Diet plan: $x_1 = \#\text{ of Hamburgers}$

     $x_2 = \#\text{ of Big Macs}$

     $x_3 = \#\text{ of Chicken McNuggets (6 pcs)}$

     $x_4 = \#\text{ of Garden Salad}$

     $x_5 = \#\text{ of Baked Apple Pies}$
2. What measure should you use to compare alternative sets of decisions?
   - Money spent on McDonald's diet
3. What restrictions limits your choices?
   - Calories obtained ≤ 2000
   - Protein obtained $\geq 55\text{g}$
   - Vitamin C obtained $\geq 100\ \%\text{ of U.S. RDA}$
   - Calcium obtained $\geq 100\ \%\text{ of U.S. RDA}$

MCDONALD'S DIET PROBLEM FORMULATION

4. Formulate the objective function:
   - $\mathbf{MIN}\ \ 0.69\,x_1 + 1.99\,x_2 + 1.99\,x_3 + 2.05\,x_4 + 0.79\,x_5$

5. Formulate the constraints:
   - Calories: $260x_1 + 560x_2 + 250x_3 + 35x_4 + 260x_5 \le 2000$
   - Protein (in g): $13x_1 + 25x_2 + 15x_3 + 2x_4 + 3x_5 \ge 55$
   - Vitamin C (in %): $2x_1 + 2x_2 + 2x_3 + 40x_4 + 40x_5 \ge 100$
   - Calcium (in %): $15x_1 + 25x_2 + 2x_3 + 4x_4 + 2x_5 \ge 100$

6. Do you need non-negativity constraints?
   - $x_i \ge 0 \text{ for } i = 1,2,3,4,5$

7. Write down the linear program:

$\mathbf{MIN}\ \ 0.69\,x_1 + 1.99\,x_2 + 1.99\,x_3 + 2.05\,x_4 + 0.79\,x_5$
s.t. (such that)
$260x_1 + 560x_2 + 250x_3 + 35x_4 + 260x_5 \le 2000$
$13x_1 + 25x_2 + 15x_3 + 2x_4 + 3x_5 \ge 55$
$2x_1 + 2x_2 + 2x_3 + 40x_4 + 40x_5 \ge 100$
$15x_1 + 25x_2 + 2x_3 + 4x_4 + 2x_5 \ge 100$
$x_i \ge 0 \text{ for } i = 1,2,3,4,5$

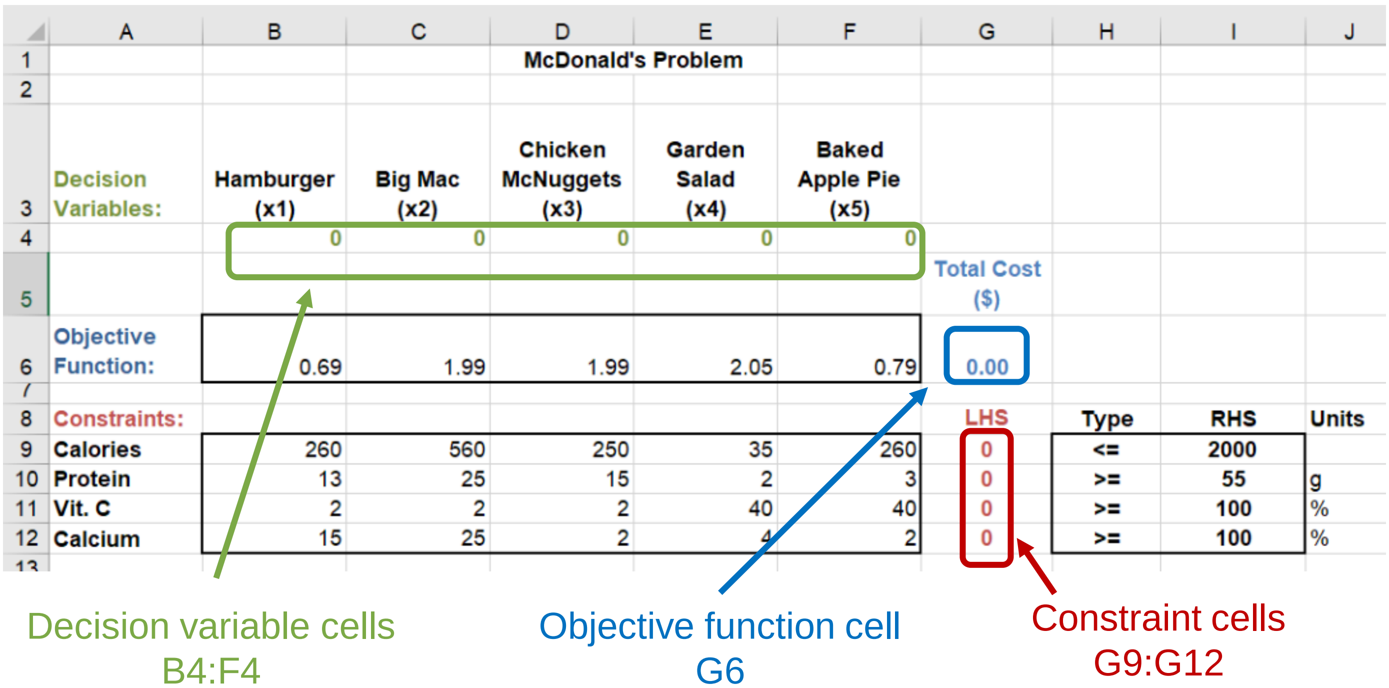
A LINEAR PROGRAM

- You just formulate McDonald’s diet problem as **a linear program.**
- The word “linear” means a directly proportional relationship.
- The word “program” means a format comprising a set of mathematical expressions.
- Linear programming (LP) is practical and useful.
- Various problems (e.g. routinely used in industry or government for planning & managing day-to-day operations) can be handled by LP.

LINEAR PROGRAMMING TERMINOLOGY

- **Decision variables**
  - The controllable variables involved in the linear program.
  - They should completely describe the decisions to be made.
  - e.g. # of burgers, etc.
- **Objective function**
  - The decision maker wants to maximize (usually revenue or profit) or minimize (usually cost) some function of decision variables. The function to be maximized or minimized is called the objective function
  - e.g. Minimize cost
- **Constraints**
  - Restrictions, target requirements (etc.) placed on a decision variable or several decision variables
  - e.g. diet restrictions

MCDONALD'S PROBLEM – SPREADSHEET SETUP

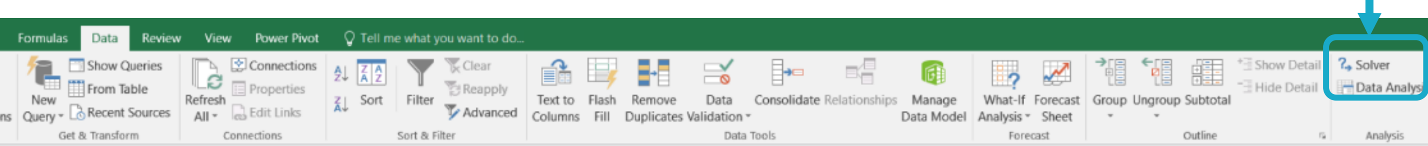


	A	B	C	D	E	F	G	H	I	J
1				McDonald's Problem						
2										
3	Decision Variables:	Hamburger (x1)	Big Mac (x2)	Chicken McNuggets (x3)	Garden Salad (x4)	Baked Apple Pie (x5)				
4		0	0	0	0	0				
5							Total Cost ($)			
6	Objective Function:	0.69	1.99	1.99	2.05	0.79	0.00			
7										
8	Constraints:						LHS	Type	RHS	Units
9	Calories	260	560	250	35	260	0	<=	2000	
10	Protein	13	25	15	2	3	0	>=	55	g
11	Vit. C	2	2	2	40	40	0	>=	100	%
12	Calcium	15	25	2	4	2	0	>=	100	%

- Enter parameters in variable cells B6:F6, B9:F12, I9:I12
- Decision variable cells:
  - Fill in zeros as initial values in cells B4:F4
- Objective function cell:
  - G6 = SUMPRODUCT(B4:F4,B6:F6)
- Constraint cells:
  - G9 = SUMPRODUCT($B$4:$F$4, B9:F9)
  - Copied to G10:G12

USE OF SOLVER

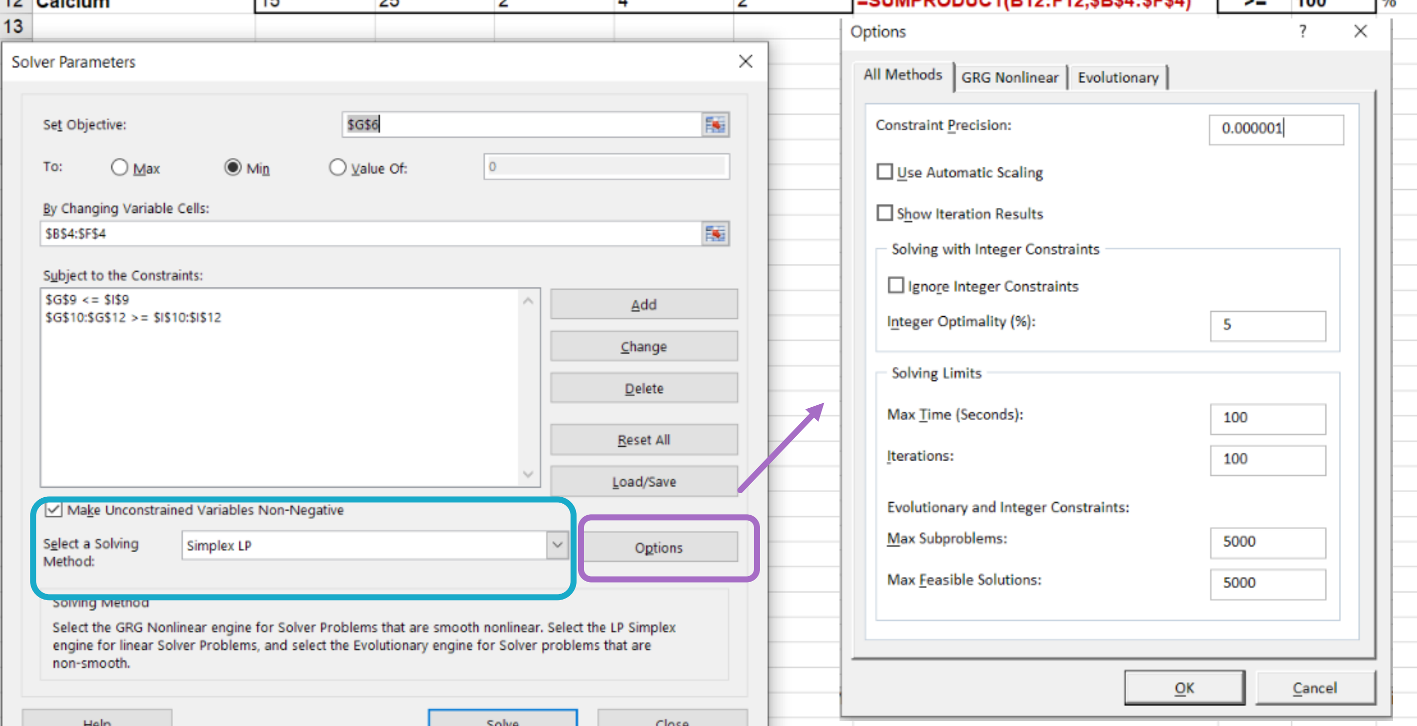
- If the Solver button does not appear on the Data tab on the Ribbon:
  1. Click File ⇒ Options ⇒ Add-Ins Category ⇒ Go.
  2. Select the Solver Add-In check box, and click OK to install it.
  3. Click Yes to confirm that you want to install the Solver add-in.



MCDONALD'S PROBLEM - USE OF SOLVER

- Data tab ⇒ Analysis ⇒ Solver.
- Objective: MIN G6
- Variables: $B4: F4$
- Constraints: $G9 \leq I9$ and $G10: G12 \geq I10: I12$
- Select a solving method: Simplex LP
- Make unconstrained variables non-negative

	A	B	C	D	E	F	G	H	I	J
1				McDonald's Problem						
2										
3	Decision Variables:	Hamburger (x1)	Big Mac (x2)	Chicken McNuggets (x3)	Garden Salad (x4)	Baked Apple Pie(x5)				
4		6.2589366956	0	0	0.870921617	1.316131548				
5							Total Cost ($)			
6	Objective Function:	0.69	1.99	1.99	2.05	0.79	=SUMPRODUCT(B4:F4,B6:F6)			
7										
8	Constraints:						LHS	Type	RHS	Units
9	Calories	260	560	250	35	260	=SUMPRODUCT(B9:F9,$B$4:$F$4)	<=	2000	
10	Protein	13	25	15	2	3	=SUMPRODUCT(B10:F10,$B$4:$F$4)	>=	55	g
11	Vit. C	2	2	2	40	40	=SUMPRODUCT(B11:F11,$B$4:$F$4)	>=	100	%
12	Calcium	15	25	2	4	2	=SUMPRODUCT(B12:F12,$B$4:$F$4)	>=	100	%
13										



ANSWER REPORTING – BINDING AND SLACK CONSTRAINTS

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$G$6	Objective Function: Total Cost ($)	0.00	7.14

Optimal objective function value

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$4	Hamburger (x1)	0.00	6.26	Contin
$C$4	Big Mac (x2)	0.00	0.00	Contin
$D$4	Chicken McNuggets (x3)	0.00	0.00	Contin
$E$4	Garden Salad (x4)	0.00	0.87	Contin
$F$4	Baked Apple Pie (x5)	0.00	1.32	Contin

Optimal decision variable values (or Optimal Solution)

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$G$10	Protein LHS	87	$G$10>=$I$10	Not Binding	32
$G$11	Vit. C LHS	100	$G$11>=$I$11	Binding	0
$G$12	Calcium LHS	100	$G$12>=$I$12	Binding	0
$G$9	Calories LHS	2000	$G$9<=$I$9	Binding	0

Constraints:

Binding means LHS = RHS, implies **Slack** = 0

Not Binding means LHS > RHS (or LHS < RHS), **Slack** = difference

ALTERNATIVE SPREADSHEET SETUP

- Build model upon the existing data structure

	A	B	C	D	E	F	G	H	I	J	K
1				McDonald's Problem							
2											
3	Total cost ($)	0									
4											
5	Menu Item	Pick	Price ($)	Calories	Protein	Fat	Sodium	Vit. A	Vit. C	Calcium	Iron
7					(g)		(mg)	% U.S.			
8	Hamburger	0	0.69	260	13	9	530	2	2	15	15
9	Big Mac	0	1.99	560	25	30	1010	8	2	25	25
10	Chicken McNuggets (6pcs)	0	1.99	250	15	15	670	2	2	2	4
11	Garden Salad	0	2.05	35	2	0	20	120	40	4	6
12	Baked Apple Pie	0	0.79	260	3	13	200	0	40	2	6
13			Consume	0	0				0	0	
14			Type	<=	>=				>=	>=	
15			RHS	2000	55				100	100	

Decision variable cells
B8:B12

Objective function cell
B3

Constraint cells
D13:E13, I13:J13

- Decision variable cells: Fill in zeros as initial values in cells B8:B12
- Objective function cell: B3 = SUMPRODUCT($B$8:$B$12, C8:C12)
- Constraint cells: D13 = SUMPRODUCT($B$8:$B$12, D8:D12)
  - Copied to E10, I13, J13
- Using Solver
  - Variables: $B8:B12$
  - Objective: MIN B3
  - Constraints: D13 $\leq D15, E13 \geq E15,$ and $I13:J13 \geq I15:J15$
  - Options: Simplex LP, make unconstrained variables non-negative

PRODUCT MIX EXAMPLE
PAR, INC. PROBLEM

- Par, Inc. manufactures two types of golf bags: **standard** and **deluxe**.
- The profit of a standard golf bag is $10. The profit of a deluxe golf bag is $9.
- The production of golf bags mainly consists of four steps: cutting & dyeing, sewing, finishing, inspection & packaging.
- Each standard golf bag requires 7/10 hour of cutting & dyeing, 1/2 hour of sewing, 1 hour of finishing, and 1/10 hour of inspection & packaging.
- Each deluxe golf bag requires 1 hour of cutting & dyeing, 5/6 hour of sewing, 2/3 hour of finishing, and 1/4 hour of inspection & packaging.
- Demand for golf bags is unlimited.
- However, due to the capacity and labour constraints, in each week, Par can perform at most 630 hours of cutting & dyeing, 600 hours of sewing, 708 hours of finishing, and 135 hours of inspection & packaging for the production of golf bags.
- Par wishes to maximize weekly profit. Formulate a linear program for Par's production problem.

PAR PROBLEM FORMULATION

1. What must be decided?

2. What measure should you use to compare alternative sets of decisions?

3. What restrictions limits your choices?

	Standard @ $10	Deluxe @ $9
Cut & dye (630 hrs)	7/10	1
Sewing (600 hrs)	1/2	5/6
Finishing (708 hrs)	1	2/3
Insp & Pckg (135 hrs)	1/10	1/4

PAR PROBLEM FORMULATION

	Std @ $10	Dlx @ $9
Cut (630)	7/10	1
Sew (600)	1/2	5/6
Fin (708)	1	2/3
Insp (135)	1/10	1/4

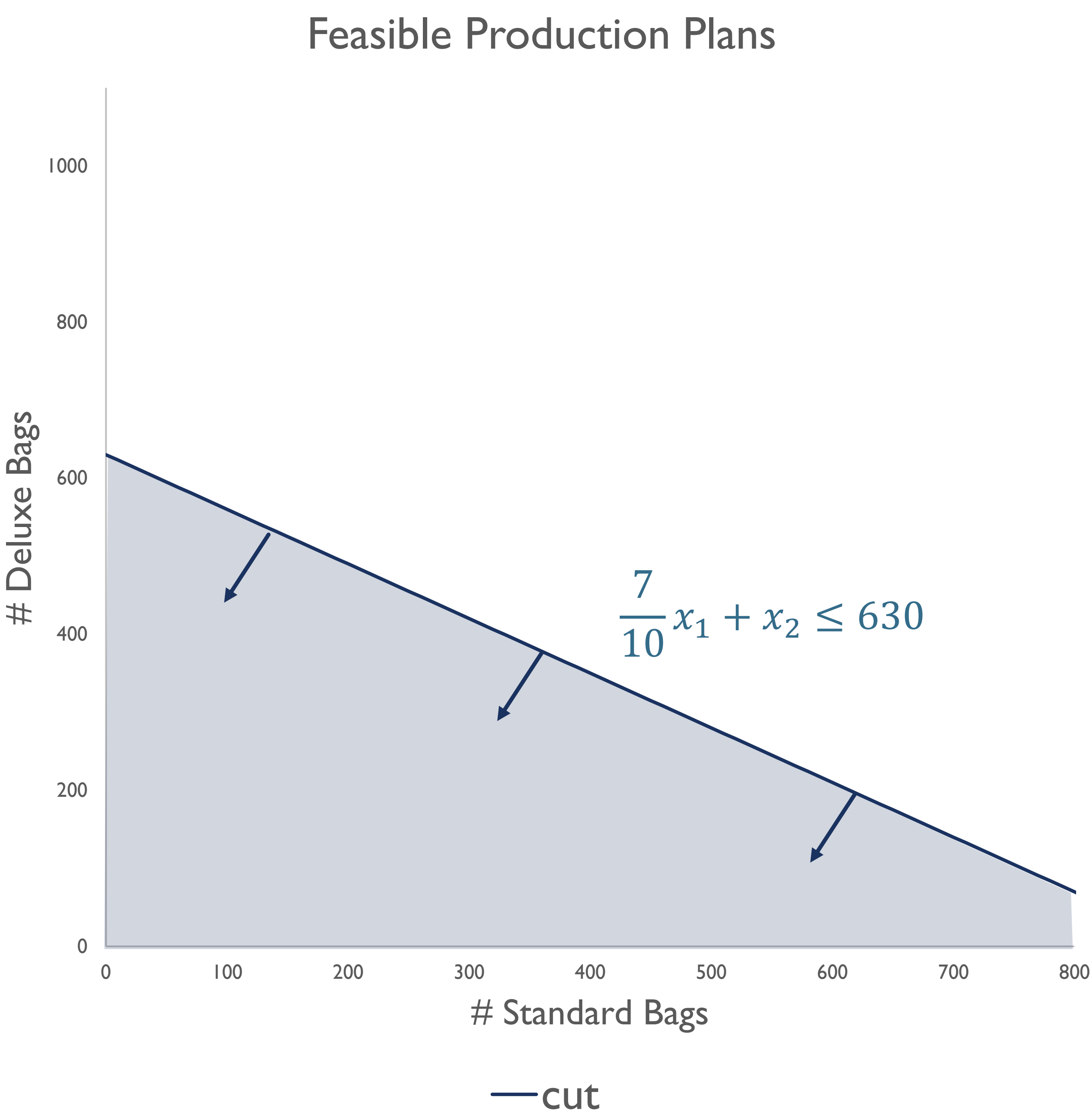
4. Formulate the objective function:

5. Formulate the constraints:

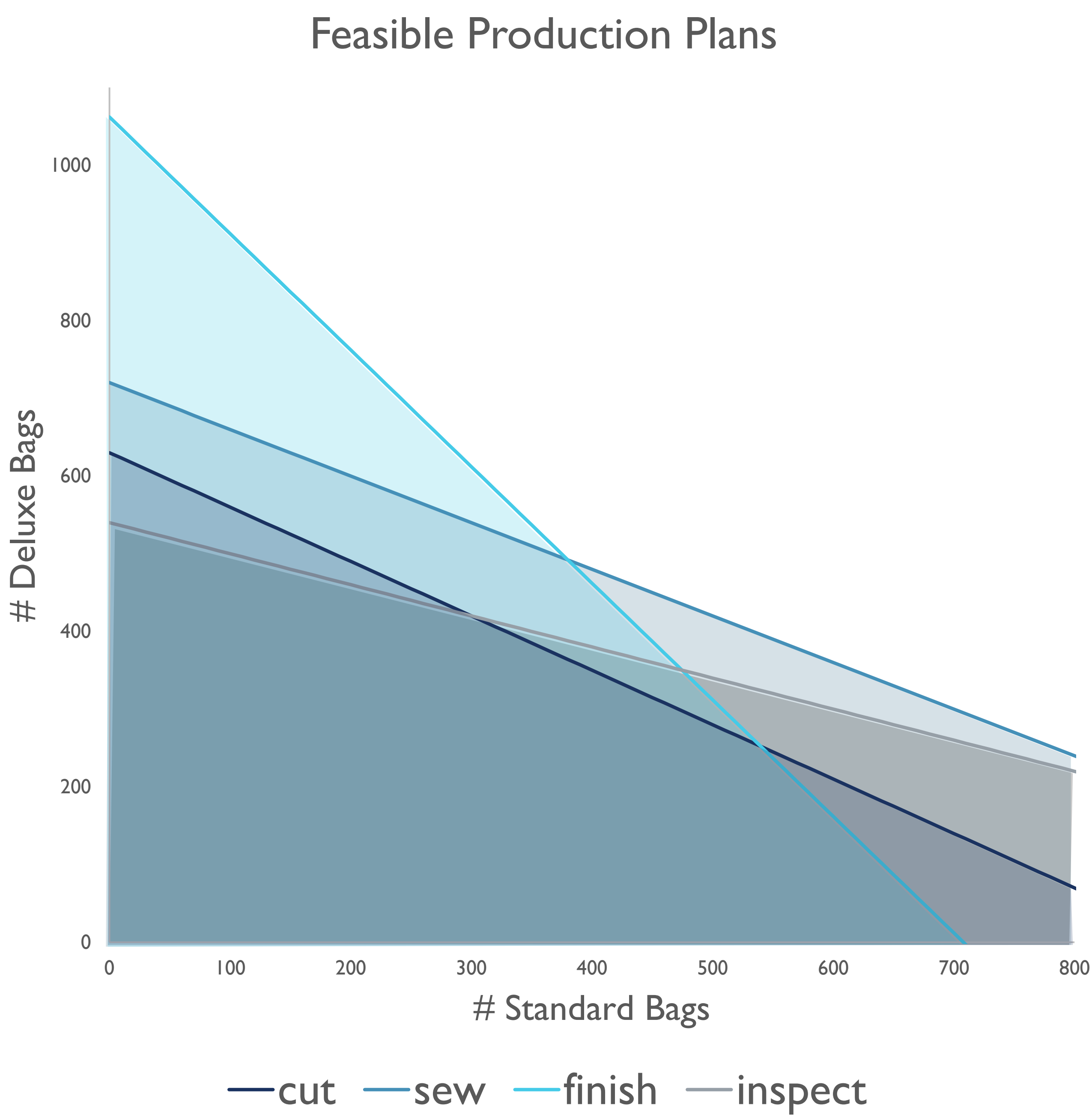
6. Do you need non-negativity constraints?

7. Write down the linear program:

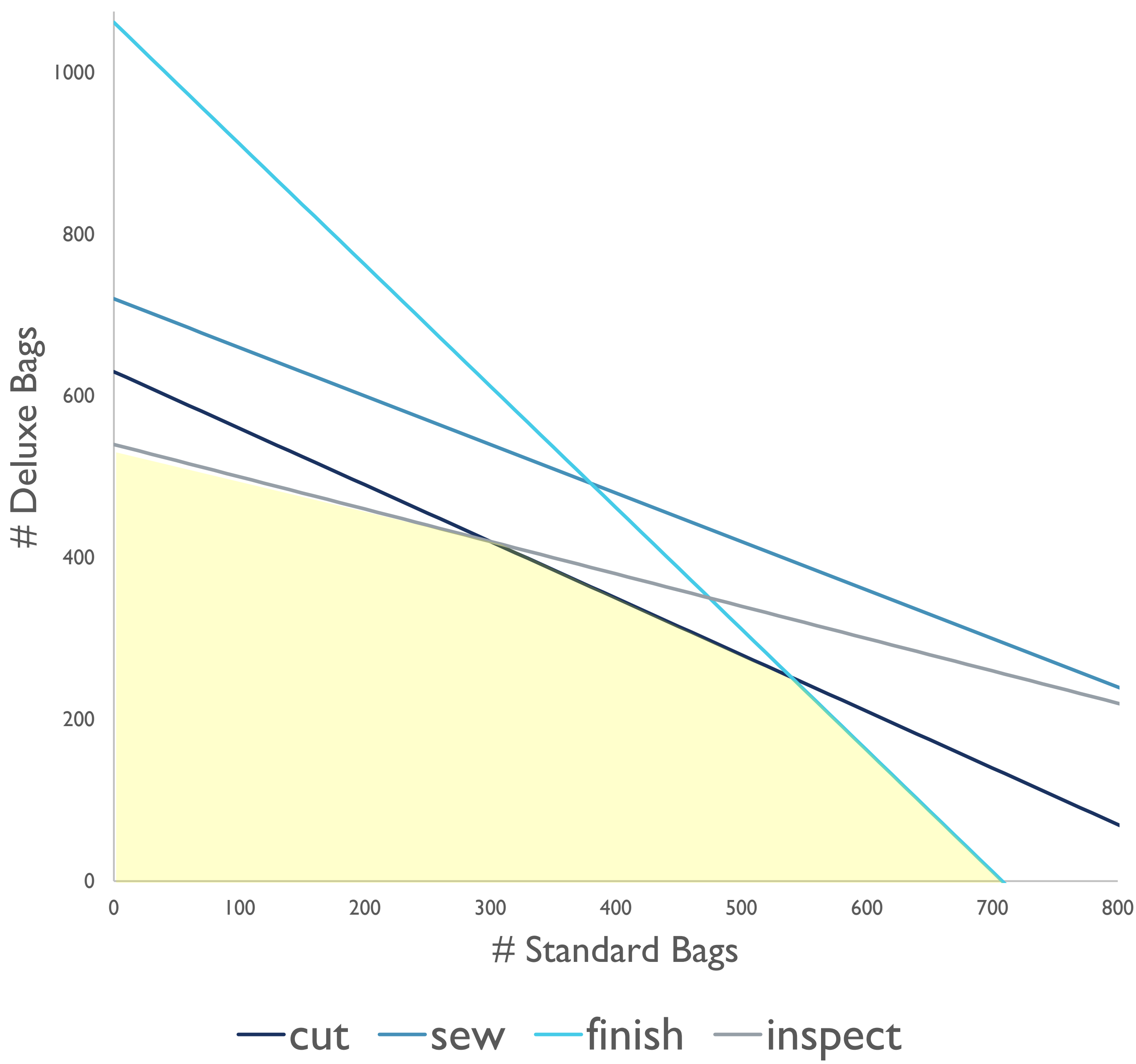
PAR PROBLEM FORMULATION



PAR PROBLEM FORMULATION



PAR PROBLEM FORMULATION



PAR PROBLEM – SPREADSHEET SETUP

	A	B	C	D	E	F	G
1		Par Inc.					
2							
3	Decision Variables	Standard Bag (x1)	Deluxe Bag (x2)				
4		0	0				
5				Total profit			
6	Objective Function:	10	9	0			
7							
8	Constraints:			LHS	Type	RHS	Units
9	Cutting & Dyeing	0.7	1	0	<=	630	Hr
10	Sewing	0.5	0.833333333	0	<=	600	Hr
11	Finishing	1	0.666666667	0	<=	708	Hr
12	Inspection & Packaging	0.1	0.25	0	<=	135	Hr

Decision variable cells B4:C4

Objective function cell D6

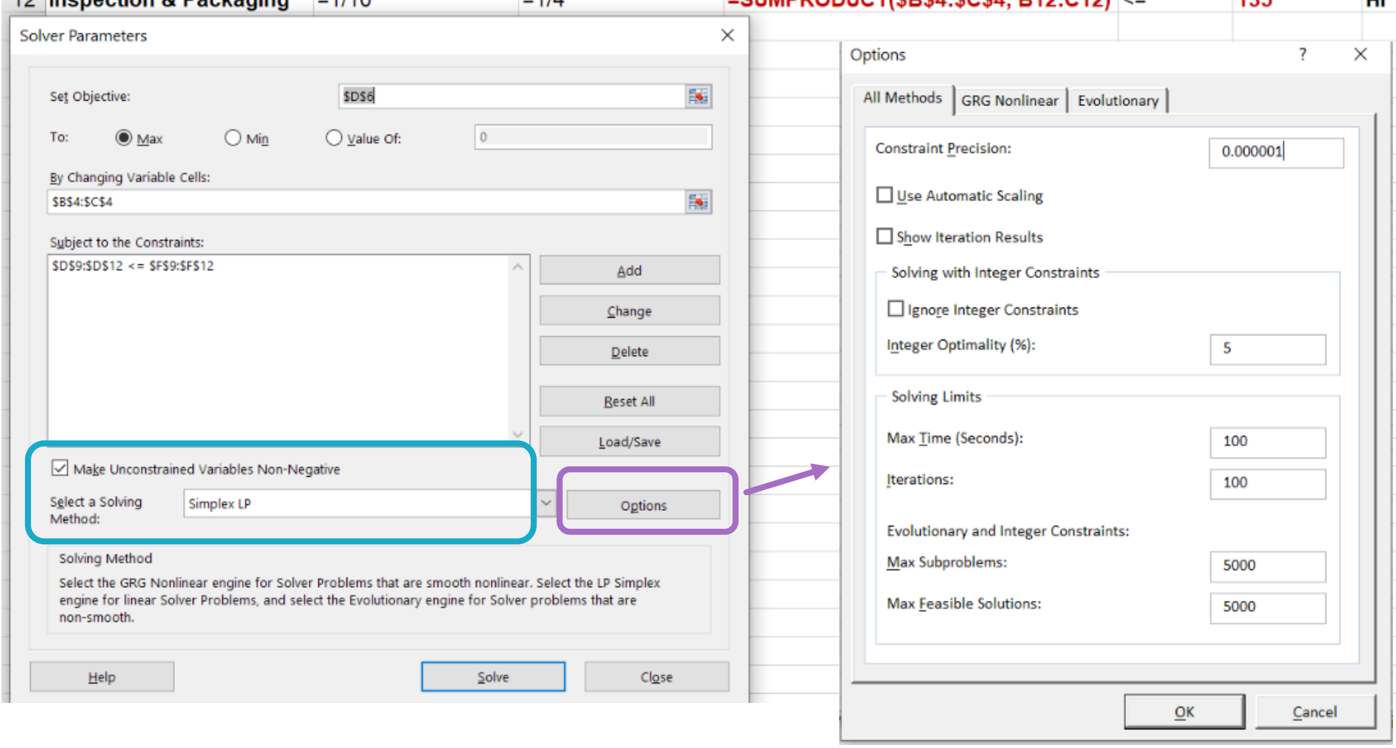
Constraint cells D9:D12

- Enter parameters in variable cells B6:C6, B9:C12, F9:F12
- Decision variable cells:
  - Fill in zeros as initial values in cells B4:C4
- Objective function cell:
  - D6 = SUMPRODUCT($B$4:$C$4,B6:C6)
- Constraint cells:
  - D9 = SUMPRODUCT($B$4:$C$4, B9:C9)
  - Copied to D10:D12

PAR PROBLEM - USE OF SOLVER

- Data tab ⇒ Analysis ⇒ Solver.
- Objective: MAX D6
- Variables: $B4: C4$
- Constraints: $D9: D12 \leq F9: F12$
- Select a solving method: Simplex LP
- Make unconstrained variables non-negative

	A	B	C	D	E	F	G
1		Par Inc.					
2							
3	Decision Variables	Standard Bag (x1)	Deluxe Bag (x2)				
4		540	252				
5				Total profit			
6	Objective Function:	10	9	=SUMPRODUCT($B$4:$C$4,B6:C6)			
7							
8	Constraints:			LHS	Type	RHS	Units
9	Cutting & Dyeing	=7/10	1	=SUMPRODUCT($B$4:$C$4, B9:C9)	<=	630	Hr
10	Sewing	=1/2	=5/6	=SUMPRODUCT($B$4:$C$4, B10:C10)	<=	600	Hr
11	Finishing	1	=2/3	=SUMPRODUCT($B$4:$C$4, B11:C11)	<=	708	Hr
12	Inspection & Packaging	=1/10	=1/4	=SUMPRODUCT($B$4:$C$4, B12:C12)	<=	135	Hr



LINEAR PROGRAMMING TERMINOLOGY (REVISIT)

- **Decision variables**
  - The controllable variables involved in the linear program.
  - They should completely describe the decisions to be made.
  - e.g. Par Inc. problem: # of standard bags to be produced, # of deluxe bags to be produced
- **Objective function**
  - The decision maker wants to maximize (usually revenue or profit) or minimize (usually cost) some function of decision variables. The function to be maximized or minimized is called the objective function
  - e.g. Par Inc. problem: maximize profit
- **Constraints**
  - Restrictions, target requirements (etc.) placed on a decision variable or several decision variables
  - e.g. Par Inc. problem: production capacities - cutting & dyeing, sewing, finishing, inspection & packaging

LINEAR PROGRAMMING TERMINOLOGY

- Linear programming (LP) is an optimization method by which you can do the following:

1. You may maximize/minimize a **linear** function of a **finite** number of decision variables.
   - Linear function: each variable associated with its coefficient appears in a separate term
   - Linear functions: i) $2x + 3y$, ii) $3.5x, 7.5y$
   - Non-linear functions: i) $13x^2 + 23xy$, ii) $\log(x) + \cos(y)$, iii) $\max(x, 0)$, iv) $if(x < 5, 0, 1)$
2. The values of the decision variables must satisfy a **finite** set of constraints. Each constraint must be a **linear** equation or a **linear** inequality.
   - $2x + 3y \ = 10$
   - $3x + 4.5y \leq 100$
   - $4x - 3.5y \geq 50$
   - Is $\frac{x}{x+y} \leq 0.4$ a linear constraint?
3. Each decision variable requires a sign restriction in order to indicate its values with practical meaning. Typically, either x_i is non-negative (i.e. $x_i > 0$ or $x_i = 0$, simply $x_i \geq 0$) or x_i is unrestricted in sign (i.e. $x_i > 0$ or $x_i = 0$ or $x_i < 0$).

EXAMPLES

- Par Problem:

$\mathbf{MAX}\ \ 10\,x_1 + 9\,x_2$ — Linear objective function

s.t.

Cutting & Dyeing: $\frac{7}{10}x_1 + x_2 \leq 630$

Sewing: $\frac{1}{2}x_1 + \frac{5}{6}x_2 \leq 600$

Finishing: $x_1 + \frac{2}{3}x_2 \leq 708$

Inspection & Packaging: $\frac{1}{10}x_1 + \frac{1}{4}x_2 \leq 135$

(Cutting & Dyeing through Inspection & Packaging: Linear constraints)

Non-negativity: $x_1 \geq 0, x_2 \geq 0$ — Sign restrictions

- General example:

$\mathbf{MAX\ (MIN)}\ \ c_1x_1 + c_2x_2 + c_3x_3 + \ \dots + c_nx_n$

s.t.

Constraint 1: $A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \ \dots + A_{1n}x_n \leq (\geq) B_1$

Constraint 2: $A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \ \dots + A_{2n}x_n \leq (\geq) B_2$

Constraint 3: $A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \ \dots + A_{3n}x_n \leq (\geq) B_3$

$\dots$

$\dots$

Constraint m: $A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 + \ \dots + A_{mn}x_n \leq (\geq) B_m$

$x_i \geq 0$ or unrestricted for $i = 1, \dots, n$

FEASIBLE AND INFEASIBLE SOLUTIONS

- Par Problem:

Objective: $\textbf{MAX}\ \ 10\,x_1 + 9\,x_2$

s.t.

Cutting & Dyeing: $\frac{7}{10}x_1 + x_2 \leq 630$

Sewing: $\frac{1}{2}x_1 + \frac{5}{6}x_2 \leq 600$

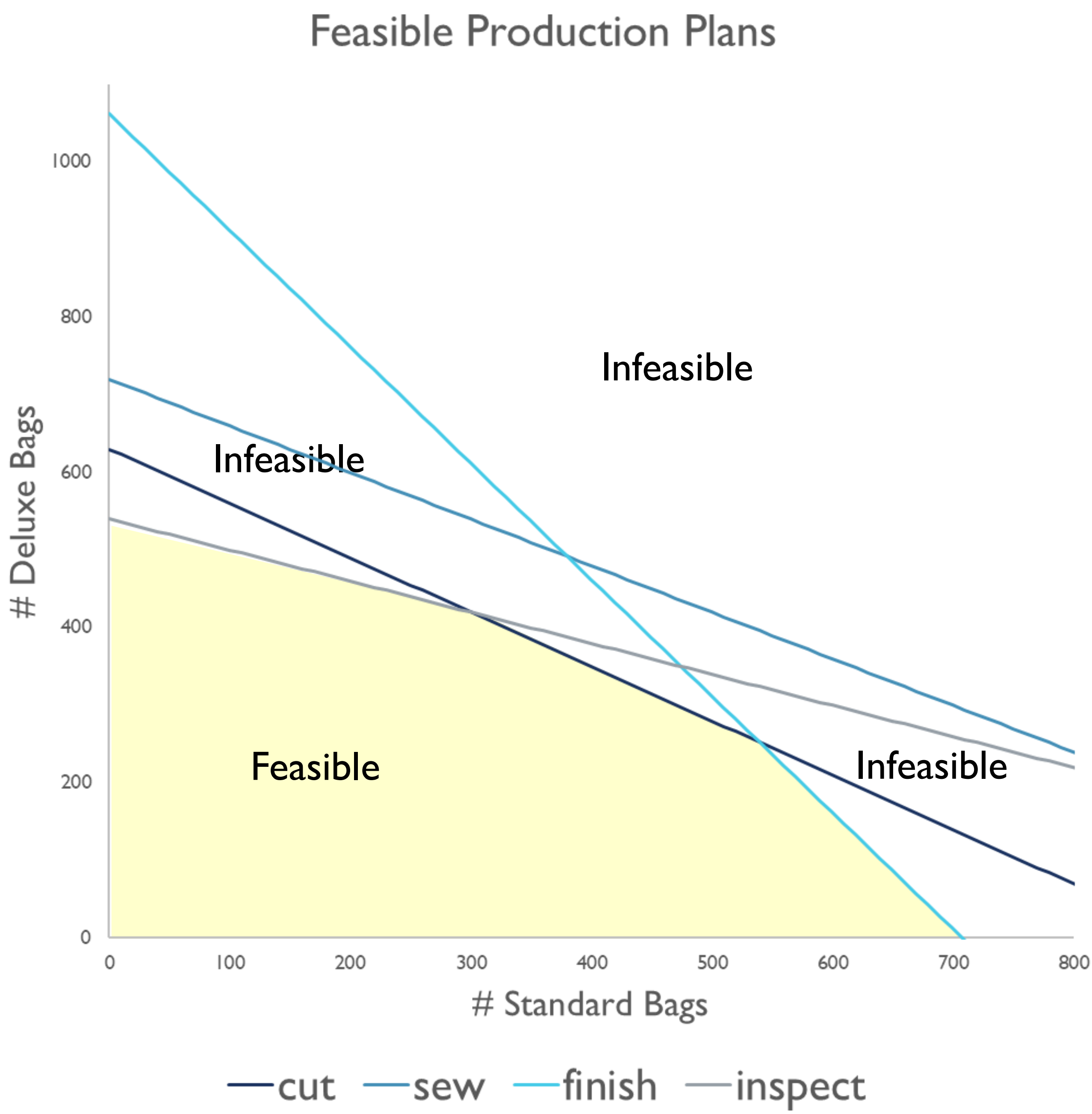
Finishing: $x_1 + \frac{2}{3}x_2 \leq 708$

Inspection & Packaging: $\frac{1}{10}x_1 + \frac{1}{4}x_2 \leq 135$

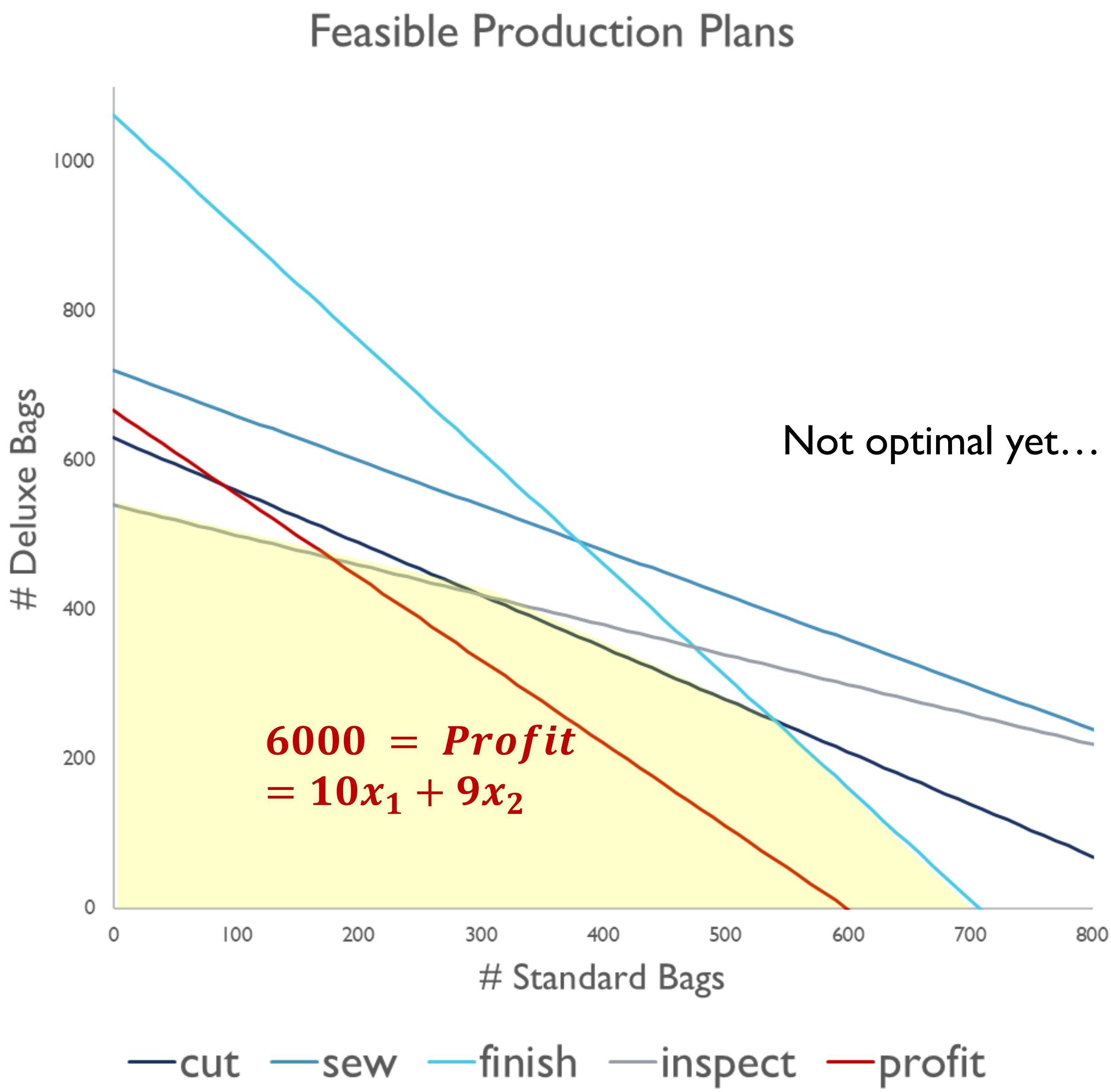
Non-negativity: $x_1 \geq 0, x_2 \geq 0$

- Feasible Solutions:
  - A production plan (x_1, x_2) that satisfies all the constraints of the problem is called a feasible solution.
  - For example, $(x_1 = 10, x_2 = 0)$ is a feasible solution.
  - Feasible region is the collection of all feasible solutions.
- Infeasible Solutions:
  - A production plan (x_1, x_2) that violates one or more constraints of the problem is called an infeasible solution.
  - For example, $(x_1 = 800, x_2 = 0)$ is an infeasible solution because it violates the constraint on “finishing”.

PAR PROBLEM FORMULATION



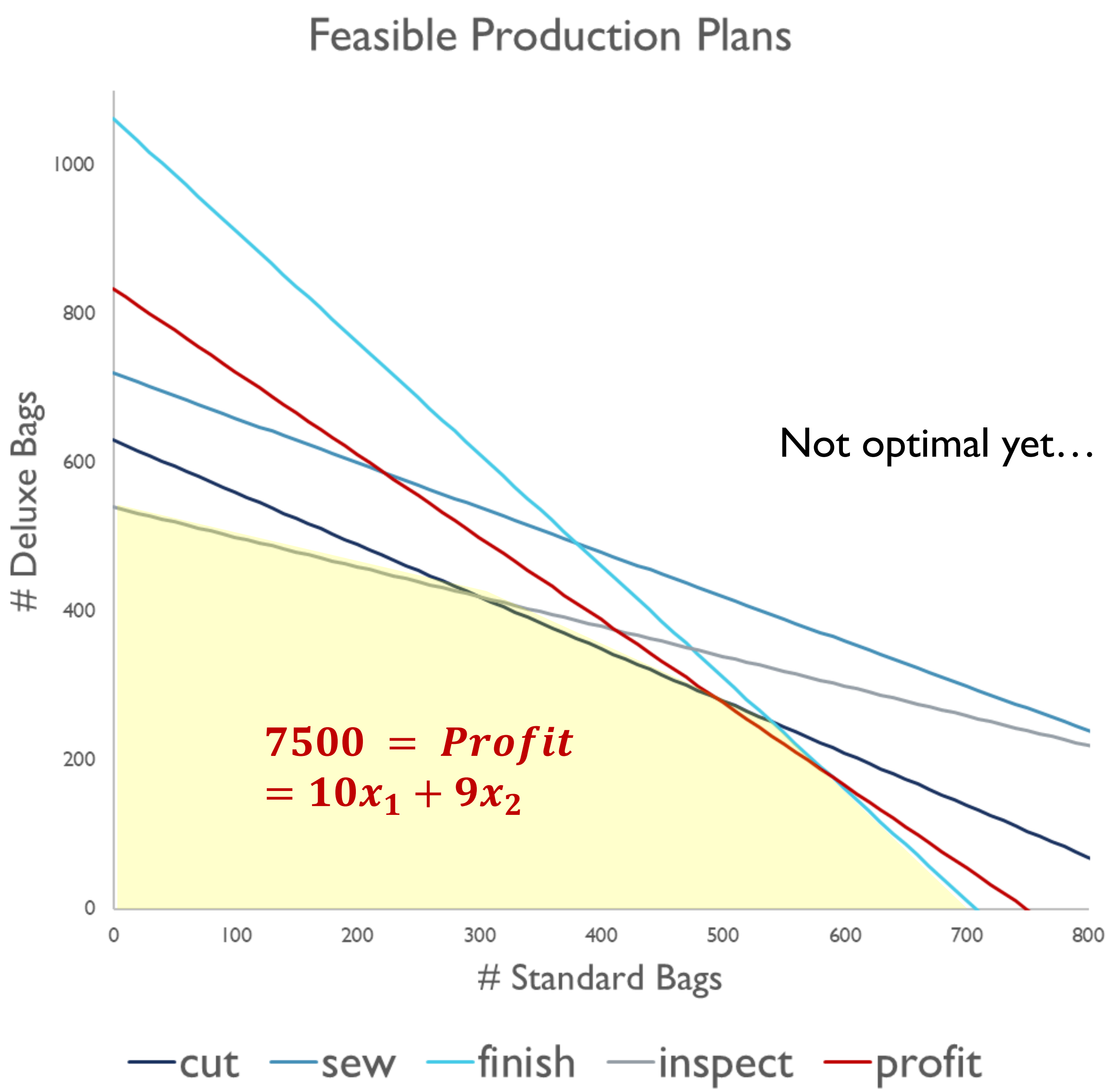
PAR PROBLEM FORMULATION



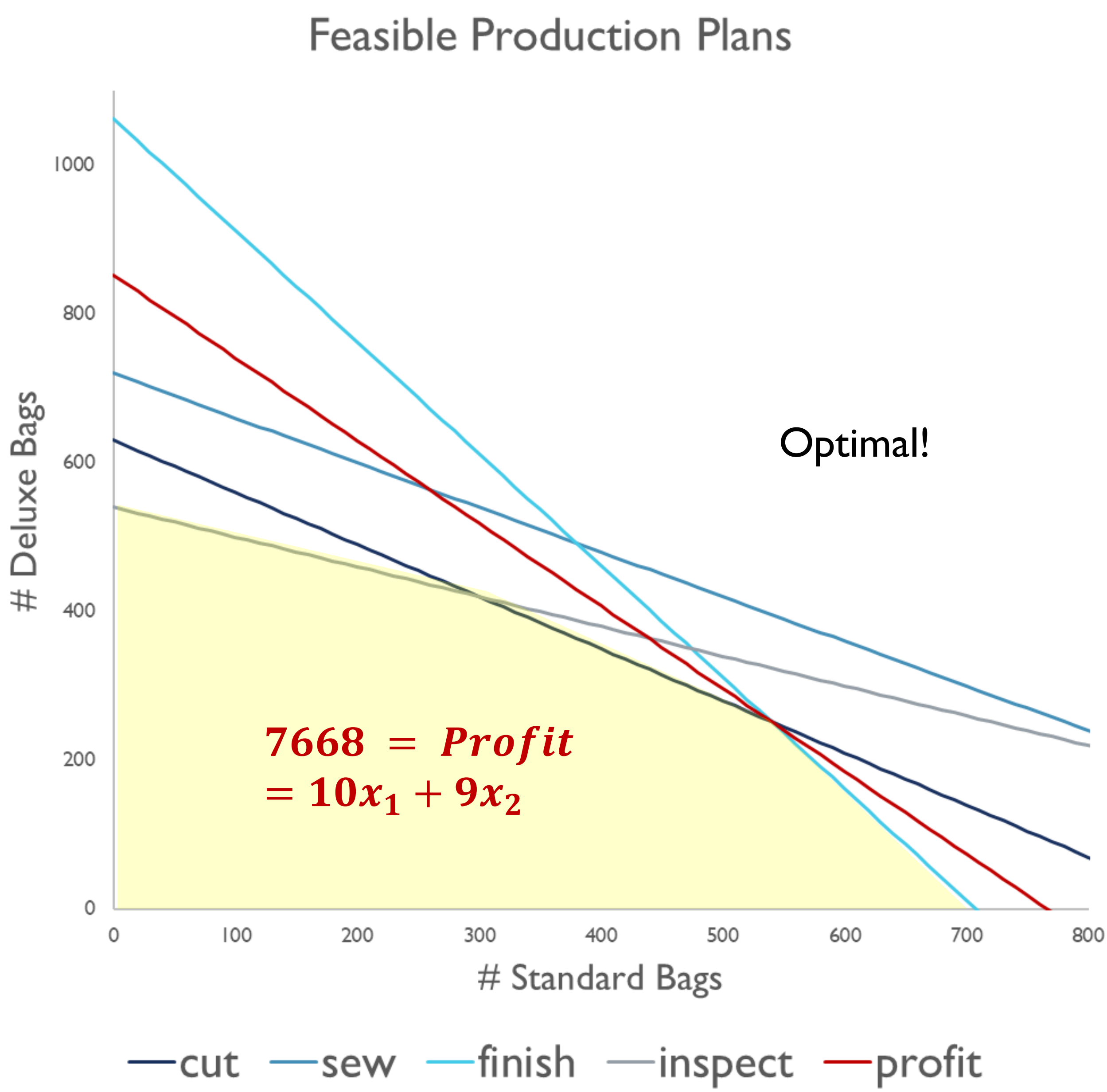
$6000 = Profit$
$= 10x_1 + 9x_2$

Not optimal yet…

PAR PROBLEM FORMULATION



PAR PROBLEM FORMULATION



OPTIMAL SOLUTIONS

- Par Problem:

Objective: $\mathbf{MAX}\ 10\,x_1 + 9\,x_2$

s.t.

Cutting & Dyeing: $\frac{7}{10}x_1 + x_2 \leq 630$

Sewing: $\frac{1}{2}x_1 + \frac{5}{6}x_2 \leq 600$

Finishing: $x_1 + \frac{2}{3}x_2 \leq 708$

Inspection & Packaging: $\frac{1}{10}x_1 + \frac{1}{4}x_2 \leq 135$

Non-negativity: $x_1 \geq 0, x_2 \geq 0$

- Optimal Solution:
  - For a maximization (or minimization) problem, the optimal solution is a feasible solution that has the largest (or smallest) objective value among all feasible solutions.
  - $(x_1 = 540, x_2 = 252)$ is the optimal solution and the maximum objective value is $\$7668$.

SPECIAL CASES OF LP SOLUTION: INCONSISTENT PROBLEM ⇒ INFEASIBLE SOLUTION

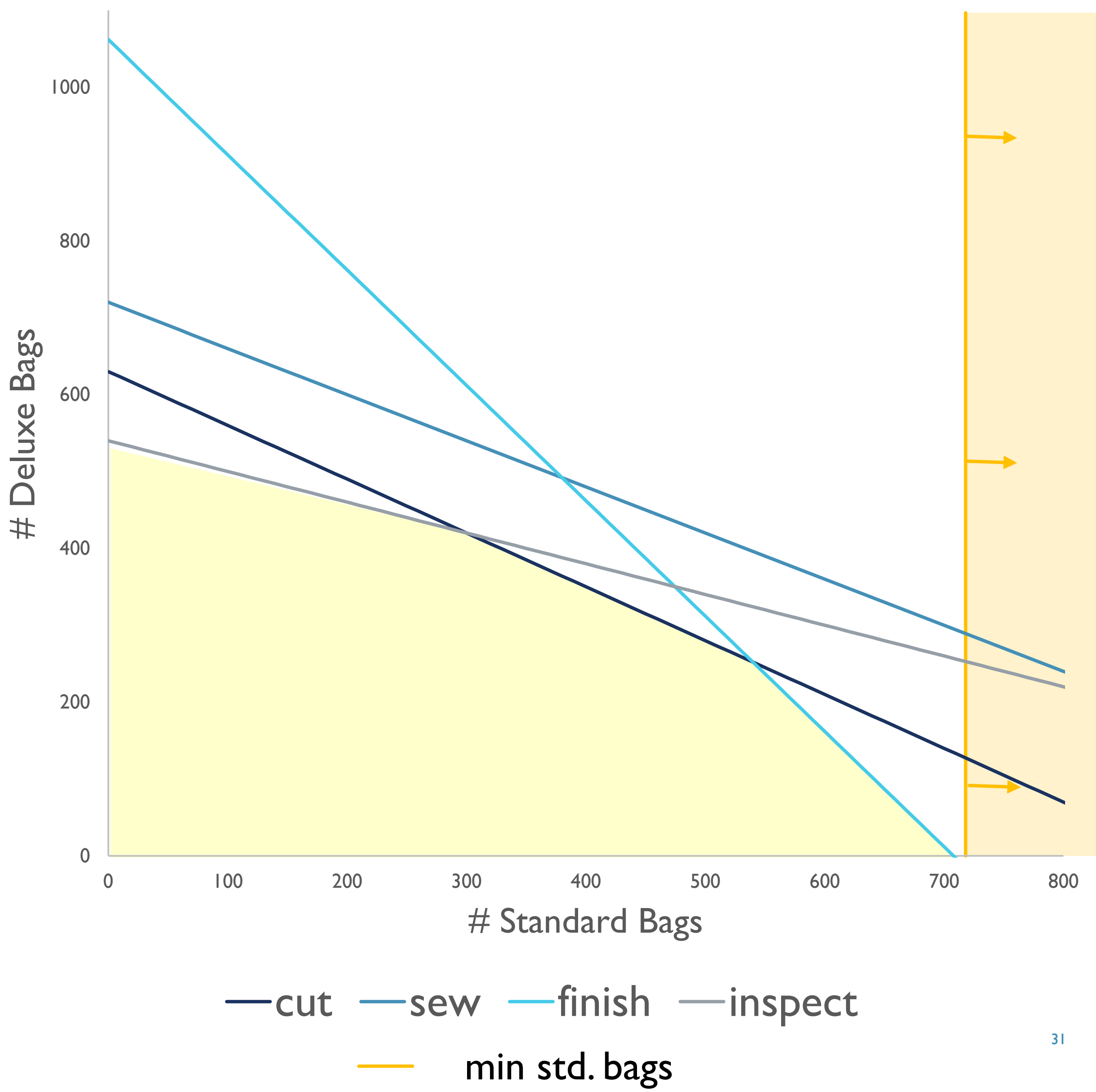
- Par Problem: Assume there is an additional constraint, and we need to produce at least 725 standard bags.

Objective:	$\textbf{MAX}\ 10\,x_1 + 9\,x_2$
s.t.	
Cutting & Dyeing:	$\frac{7}{10}x_1 + x_2 \le 630$
Sewing:	$\frac{1}{2}x_1 + \frac{5}{6}x_2 \le 600$
Finishing:	$x_1 + \frac{2}{3}x_2 \le 708$
Inspection & Packaging:	$\frac{1}{10}x_1 + \frac{1}{4}x_2 \le 135$
	$x_1 \ge 725$
Non-negativity:	$x_1 \ge 0, x_2 \ge 0$

- The new constraint, $x_1 \ge 725$, contradicts (or is inconsistent) with the constraint on "finishing". See plot on next slide.
- We may also say that the problem is **infeasible**.

PAR PROBLEM FORMULATION

No feasible production plans exist



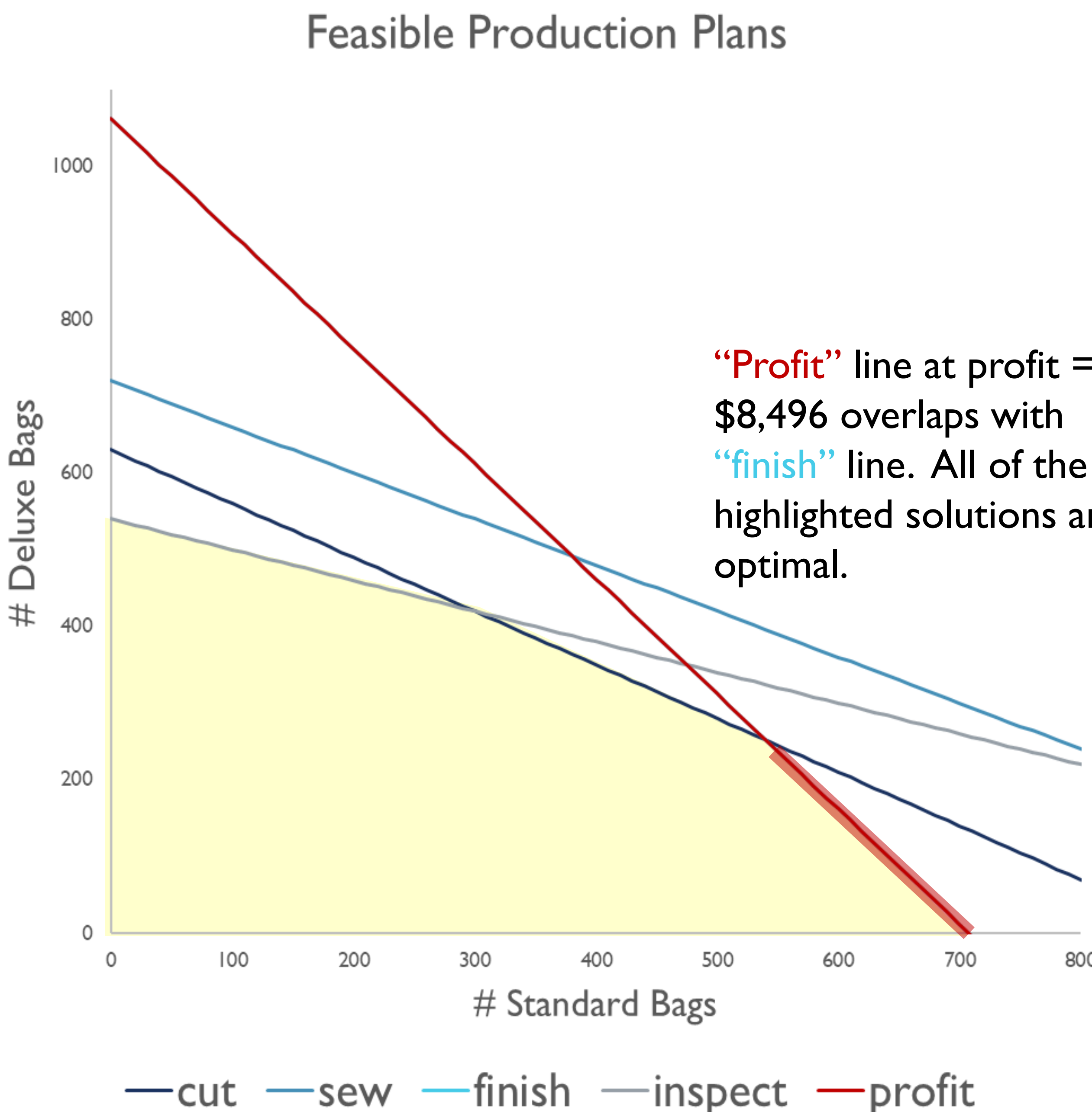
SPECIAL CASES OF LP SOLUTION: MULTIPLE OPTIMAL SOLUTIONS PROBLEM

- Par Problem: Assume the profit of the standard bag is $\$12$ and the profit of the deluxe bag is $\$8$.

Objective:	$\mathbf{MAX}\ 12\,x_1 + 8\,x_2$	Slope of line?
s.t.		
Cutting & Dyeing:	$\frac{7}{10}x_1 + x_2 \le 630$	
Sewing:	$\frac{1}{2}x_1 + \frac{5}{6}x_2 \le 600$	
Finishing:	$x_1 + \frac{2}{3}x_2 \le 708$	Slope of line?
Inspection & Packaging:	$\frac{1}{10}x_1 + \frac{1}{4}x_2 \le 135$	
Non-negativity:	$x_1 \ge 0, x_2 \ge 0$	

- More than one optimal solution exists.
- Both $(x_1 = 540, x_2 = 252)$ and $(x_1 = 708, x_2 = 0)$ yield the same profit of $\$8496$.
- In fact, there are many optimal solutions (see next slide).

PAR PROBLEM FORMULATION



"Profit" line at profit = $8,496 overlaps with "finish" line. All of the highlighted solutions are optimal.

SPECIAL CASES OF LP SOLUTION: UNBOUNDED PROBLEM

- Consider the following problem:

$$\textbf{MAX} \;\; 12\,x_1 + 8\,x_2$$

s.t.

$$x_1 + x_2 \geq 1000$$
$$x_2 \geq 725$$
$$x_1 \geq 0, x_2 \geq 0$$

- What does the diagram look like?

- x_1 and x_2 can take values as large as we want, and neither has an upper-bound.

SUMMARY OF SPECIAL CASES

Special Cases	**Graphical Representation**	**Excel Solver Message**	**Managerial Implication**
Inconsistent problem	Feasible solution does not exist.	"Solver could not find a feasible solution".	Too many restrictions.
Multiple optimal solutions	The slope of the objective function is the same as the slope of one of the constraints.	Give one optimal solution.	How do we detect multiple optimal solutions?
Unbounded problem	Optimal objective function value is infinite.	"The Objective Cell values do not converge".	Problem is improperly formulated: -too few constraints, or -wrong objective function.

LP FORMULATION GUIDELINES

- Formulation (also called modeling): The process of translating a verbal statement of a problem into a mathematical statement.
- Guidelines for formulation:

1. Understand the problem.
2. Ask the following three questions:
   i. What must be decided? What are the decision variables?
   ii. What measure should you use to compare alternative sets of decisions?
   iii. What restrictions limit your choices?
3. Write the mathematical representation of the objective function in terms of the decision variables.
4. Write the constraints in terms of the decision variables.
5. Check your formulation!
   - Does it make sense?
   - Are there any data missing in the problem?